When Do the Moments Uniquely Identify a Distribution

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Abstract

In this brief note we show that whenever X is a positive random variable, if $E(X^h)$ is defined for h in some neighborhood of zero then the moments $E(X^h)$ uniquely identify the distribution of X.

Key words: moment problem, identifiability of a distribution through its moments, separating functions, Log-Normal distribution, F distribution.

1. INTRODUCTION

We know that if a random variable (r.v.) X has a bounded support or if it has an unbounded support but it has a moment generating function (m.g.f.) then the set of its positive integer order moments $E(X^h)$, $h \in I\!N$, identifies uniquely such r.v.. In this latter case, its m.g.f. $M_X(t)$, being analytic may be expressed as

$$M_X(t) = \sum_{i=0}^{\infty} t^i \, E(X^i)/i! \qquad \text{for } t \in V_{\epsilon,\epsilon'}(0)\,,$$

where $V_{\epsilon,\epsilon'}(0) = \{x : x \in]-\epsilon, \epsilon'[,\ \epsilon, \epsilon' > 0\}$ represents a neighborhood of zero (Chung, 1974; Feller, 1971). For r.v.'s that although not having a m.g.f. still have positive integer moments of all orders, despite the fact that we may think about checking for the applicability of the Carleman conditions (Carleman, 1926) or Carleman type conditions (Akhiezer, 1965; Lin, 1997; Wu, 2002), we will be faced with the fact that quite often the conditions for their applicability are equivalent to ask for at least the analyticity of the characteristic function (c.f.) around $t = 0$ (Feller, 1971; Shiryaev, 1996) (see Appendix A for some more details), so that we are left with the case of the r.v.'s that do not have a c.f. that is analytic around $t = 0$ and/or do not have integer moments of all positive integer orders. However, it is possible to establish that for some such variables, the set of all their moments uniquely identify that r.v., as it is the case for example with the Log-Normal and F distributed r.v.'s. This is the topic of this brief note.

Let us start with an example which, under some aspects, is a rather well known one. Let us consider the sequence of r.v.'s X_n, $(n \in I\!N_0)$, with p.d.f.^{s} (probability density functions)

$$f_{X_n}(x) = \frac{e^{-\frac{1}{2}(\log x)^2}}{\sqrt{2\pi}\, x} \left\{1 + \alpha \sin(2n\pi \log x)\right\}, \quad |\alpha| \le 1 \tag{1}$$

which for $n = 0$ and/or $\alpha = 0$ yields the well known standard Log-Normal distribution.

It is a quite well known fact that for any $n \in I\!N_0$,

$$E\left(X_n^h\right) = e^{h^2/2}\,, \quad \forall h \in I\!N_0\,, \tag{2}$$

which is neither a function of n nor of α (see for example Feller (1971), Knight (2000) and also Casella and Berger (2002), for the cases $n = 0$ and $n = 1$), a problem and an example actually brought to our attention by Stieltjes himself (Stieltjes, 1894, 1895, Ch. VIII).

However, in the usual definition of $E\left(X^h\right)$ (which using the notation of the Stieltjes integral is defined as $E\left(X^h\right) = \int_S x^h \, dF_X(x)$ for all cases where $\int_S |x|^h \, dF_X(x)$ is convergent, where S is the support of the r.v. X and $F_X(x)$ its cumulative distribution function) nothing goes against to consider $h \in I\!R$ and indeed we should note both that (2) is still valid for any h such that $2nh \in \mathbb{Z}$, while it is

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no longer valid for $2nh \in I\!R\backslash \mathbb{Z}$. Indeed, for any $h \in I\!R$ we have

$$
\begin{aligned}
E\left(X_n^h\right) &= \int_0^\infty x^h \, \frac{e^{-\frac{1}{2}(\log\, x)^2}}{\sqrt{2\pi}\, x} \, \{1+\alpha\, \sin(2n\pi\, \log\, x)\}\, dx \\
&= e^{h^2/2} \underbrace{\int_0^\infty \frac{e^{-\frac{1}{2}(\log\, x-h)^2}}{\sqrt{2\pi}\, x} \, \{1+\alpha\, \sin(2n\pi\, \log\, x)\}\, dx}_{=1+e^{-\frac{1}{2}(2n\pi)^2}\,\alpha\,\sin(2n\pi h)} \\
&= e^{h^2/2} \left\{1+e^{-\frac{1}{2}(2n\pi)^2} \alpha\, \sin(2n\pi h)\right\},
\end{aligned}
\tag{3}
$$

where the details in the computation of the integral were left aside since they are a bit lengthy while not being the key point in this brief note.

Expression (3) clearly shows that $E\left(X_n^h\right)$ is indeed a function of both n and α, as long as $2nh \notin \mathbb{Z}$ (i.e., $2nh \in I\!R\backslash\mathbb{Z}$), what is usually an overlooked detail. Moreover, in this case we may even say that although for $h \in \mathbb{Z}$, $E\left(X_n^h\right)$ is given by (2), thus being neither a function of n nor of α, for $h \in I\!R$ the moments $E\left(X_n^h\right)$ uniquely identify the distributions in (1).

Then the pertinent question is: 'Why is that and when is it the case that the moments $E\left(X^h\right)$, for $h \in C \subseteq I\!R$, uniquely identify the distribution of X and how is the set C defined?'.

2. THE MAIN RESULTS

In simple terms, the function $w(\cdot)$ is said to be separating if and only if for any two c.d.f.s (cumulative distribution functions) $F(\cdot)$ and $G(\cdot)$,

$$\int_{I\!R} w(x)\, dF(x) = \int_{I\!R} w(x)\, dG(x) \implies F(x) = G(x),\ \forall x \in I\!R\,.$$

First of all let us state that, for $i = \sqrt{-1}$, and $h \in I\!R$, the functions x^{ih} are always separating for the distributions of r.v.'s with support $I\!R^+$, since if the r.v. has support $I\!R^+$ we may always define the r.v. $Y = \log\, X$, with

$$E\left(X^{ih}\right) = E\left(e^{ih\, \log\, X}\right) = E\left(e^{ihY}\right) = \Phi_Y(h)\,, \quad h \in I\!R\,, \tag{4}$$

what shows that the moments $E\left(X^{ih}\right)$, with $h \in I\!R$, will always exist and will uniquely identify the distribution of X, provided that X has support $I\!R^+$, since then there will be a r.v. $Y = \log\, X$, which will have c.f. $\Phi_Y(h) = E\left(X^{ih}\right)$. Then, using a similar argument and since the m.g.f. of Y, when it exists, also uniquely identifies the distribution of Y, we may say that if $M_Y(h) = E\left(e^{hY}\right)$ exists, similarly to (4),

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t\, \log\, X}\right) = E\left(X^t\right)\,, \quad t \in C \subseteq I\!R$$

so that if X is a r.v. so that the r.v. $Y = \log\, X$ is defined and it has a m.g.f. for $t \in C = V_{\epsilon,\epsilon'}(0)$, with

$$V_{\epsilon,\epsilon'}(0) \;=\; \left\{ x \in I\!R:\ x \in]-\epsilon, \epsilon'[\,;\ \epsilon, \epsilon' \in \overline{I\!R}^+ \right\}$$

where $\overline{I\!R}^+ = I\!R^+ \cup \{+\infty\}$, with $V_{-\infty,\infty}(0) = I\!R$.

In the case of our starting example, in fact the r.v.'s $Z_n = \log\, X_n$ have p.d.f.s

$$f_{Z_n}(z) \;=\; \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, (1+\alpha\, \sin(2n\pi z))$$

with m.g.f.s given by

$$
\begin{aligned}
M_{Z_n}(t) &= E\left(e^{tZ_n}\right) \;=\; E\left(X_n^t\right) \\
&= e^{t^2/2}\left(1+e^{-\frac{1}{2}(2n\pi)^2}\,\alpha\, \sin(2n\pi t)\right), \quad t \in I\!R
\end{aligned}
\tag{5}
$$

so that the functions x^t with $t \in I\!R$ are separating for the distributions of the r.v.'s X_n and the moments $E\left(X_n^h\right)$, for $h \in I\!R$, in this case, uniquely identify the distributions of the r.v.'s X_n, for $n \in I\!N_0$ (see Appendix B for some numerical issues).

Another example would be the F distribution with, say, m and n degrees of freedom. We know that if

$$X \sim F_{m,n}$$

then

$$E\left(X^h\right) = \frac{\Gamma\left(\frac{m}{2}+h\right)\Gamma\left(\frac{n}{2}-h\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^h, \quad -\frac{m}{2} < h < \frac{n}{2}$$

where we should notice that the moments are defined for any real h in the range specified above. This implies that the r.v. $Y = \log X$ has m.g.f.

$$M_Y(t) = \frac{\Gamma\left(\frac{m}{2}+t\right)\Gamma\left(\frac{n}{2}-t\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{m}{n}\right)^h, \quad -\frac{m}{2} < t < \frac{n}{2}$$

and thus the $F_{m,n}$ distribution is uniquely identified by its moments of order h with $-\frac{m}{2} < h < \frac{n}{2}$. We know that for $0 < n \leq 2$ the $F_{m,n}$ distribution does not even has an expected value, but this has only a psychological effect on us, since it is yet characterized by its moments of order $-\frac{m}{2} < h < n/2 \leq 1$.

A fact sometimes overlooked is that we may prove that if, for a given r.v. X, $E(X^h)$ exists (and is finite) for some $h > 0$, then it exists for any $r \in \,]0, h]$ and if $E(X^h)$ exists (and is finite) for some $h < 0$ then it also exists for any $r \in [h, 0[$. Indeed we may state the following result.

Theorem 1 *If the r.v. $X > 0$ has $E(X^h)$ defined for some $h > 0$ and some $h < 0$ then $E(X^h)$ characterizes uniquely this r.v. for the whole range of values of h for which these expected values are defined.*

Proof: First of all we should consider that if $E\left[(g(X))^h\right]$ exists (and is finite) for some $h > 0$, that is, if

$$\int_S |g(x)|^h \, dF(x) \tag{6}$$

where S is the support of the r.v., converges for some $h > 0$, then also

$$\int_S |g(x)|^r \, dF(x) \tag{7}$$

converges for any $0 < r < h$ and if (6) converges for some $h < 0$, then also (7) converges for any $h < r < 0$. In both cases we may write

$$E\left[|g(X)|^r\right] \leq P\left(|g(X)|^r \leq 1\right) + E\left[|g(X)|^h\right] \quad (< \infty),$$

since for $0 < r < h$ we may write

$$\begin{aligned} & E\left[|g(X)|^r\right] \\ & = \int_{\{x:|g(x)|^r \leq 1\}} |g(x)|^r \, dF(x) + \int_{\{x:|g(x)|^r > 1\}} |g(x)|^r \, dF(x) \\ & \leq \int_{\{x:|g(x)|^r \leq 1\}} 1 \, dF(x) + \int_{\{x:|g(x)|^r > 1\}} |g(x)|^h \, dF(x) \\ & \leq P\left[|g(X)|^r \leq 1\right] + E\left[|g(X)|^h\right] \end{aligned} \tag{8}$$

while for $h < r < 0$ we have

$$\begin{aligned} |g(x)|^r > 1 \quad & \Longleftrightarrow \quad |g(x)|^{-h} < |g(x)|^{-r} < 1 \\ & \Longleftrightarrow \quad |g(x)|^h > |g(x)|^r \; (> 1) \end{aligned}$$

and thus, following similar steps to the ones in (8), we have a similar result for $h < r < 0$.

But then we will also have for $Y = \log X$ its m.g.f. given by

$$\begin{aligned} M_Y(h) &= E\left[e^{hY}\right] = E\left[e^{h \log X}\right] \\ &= E\left[X^h\right], \qquad h \in V_{\epsilon,\epsilon'}(0), \end{aligned}$$

so that in this case the r.v. $Y = \log X$ has for sure a m.g.f., being as such uniquely characterized by $E\left[e^{hY}\right]$ and as such also the distribution of $X = e^Y$, being uniquely defined as a function of Y, is also uniquely characterized by $E\left[e^{hY}\right] = E\left[X^h\right]$. ■

3. CONCLUSIONS

As a conclusion we may say that the answer to the question 'When are the functions x^h $(h \in C \subset I\!\!R)$ separating or, equivalently, when do the moments $E\left(X^h\right)$ $(h \in C \subset I\!\!R)$ uniquely identify the distribution of X and how is the set C defined?' is:

i) whenever X has support $I\!\!R^+$, being thus possible to define the r.v. $Y = \log X$

ii) and, at the same time, the r.v. $Y = \log X$ has a m.g.f. $M_Y(t) = E\left(X^t\right)$ for $t \in C =]-\epsilon, \epsilon'[= V_{\epsilon,\epsilon'}(0)$, $(\epsilon, \epsilon' > 0)$, or equivalently, the r.v. X has $E(X^t)$ defined for $t \in C$;

so that we may state that

- if X_1 and X_2 are two r.v.'s with support $I\!\!R^+$ such that $E\left(X_1^h\right) = E\left(X_2^h\right)$ for any $h \in V_{\epsilon,\epsilon'}(0)$, where $V_{\epsilon,\epsilon'}(0)$ is the set of all values h for which $E\left(X_1^h\right)$ is defined, then the r.v.'s X_1 and X_2 are the same (the reason being that if $E\left(X_1^h\right)$ exists for any $h \in V_{\epsilon,\epsilon'}(0)$, then the m.g.f. of $Y_1 = \log X_1$ exists for any $h \in V_{\epsilon,\epsilon'}(0)$, since for $h \in V_{\epsilon,\epsilon'}(0)$

$$E\left(X_1^h\right) = E\left(e^{h \log X_1}\right) = E\left(e^{hY_1}\right) = M_{Y_1}(h),$$

and thus the m.g.f. of $Y_1 = \log X_1$ and $Y_2 = \log X_2$ are the same and thus Y_1 and Y_2 are the same r.v., and thus also X_1 and X_2 are the same r.v. since the logarithm is a one-to-one transformation);

or we may also just say that

- if X has support $I\!\!R^+$ and $E(X^h)$ is defined both for some $h > 0$ and some $h < 0$, then $E(X^h)$, for $h \in C$ uniquely identifies the distribution of X, where C is the set of values h for which $E(X^h)$ is defined, or equivalently the set of values h for which $M_{-\log X}(h)$ is defined; that is, the expression for the moments $E(X^h)$ uniquely identifies the r.v. X (in the same way that the expression for $M_Y(h)$, with $Y = -\log X$, uniquely identifies the distribution of Y);

and thus we may say that

- x^h, with $h \in V_{\epsilon,\epsilon'}(0)$, where $V_{\epsilon,\epsilon'}(0)$ is the set of all values h for which $E\left(X^h\right)$ is defined (that is, the set of all values h for which $\int_S |x^h|\, dF_X(x)$ converges), is separating for the distributions of r.v.'s in $I\!\!R^+$.

As we saw, when we consider the real order moments, these moments characterize the distribution they come from in a more insightful way then the integer order moments since, as for example Ortigueira *et all.* (2004) and Machado (2003) state, "integer-order derivatives depend only on the local behavior of a function, while fractional order derivatives depend on the whole history of the function". If we are concerned with the fact that it may seem not possible to obtain such moments from the c.f. or the m.g.f. as we usually do with the integer order moments, we should note that we only have to consider the real order derivatives of those functions in order to get it working and that such real order derivatives if taken as the extension for $h \in I\!\!R$ of

$$\left.\frac{\partial^h}{\partial t^h}\,\Phi_X(t)\right|_{t=0} = i^h \int_S x^h\, dF_X(x),$$

will not only agree with the usual Grünwald-Letnikov definition of non-integer order derivative (Miller and Ross, 1993; Samko *et all.*, 1987) as well as with other more general definitions as the Cauchy convolutional definition of derivative in Ortigueira *et all.* (2004).

APPENDIX A

A brief note on the Carleman and other Carleman type conditions

The aim of this Appendix is to briefly refer the usual Carleman conditions and to briefly summarize some of the newer Carleman type conditions obtained by some authors. For the proofs we refer the reader to the references cited.

A stronger result than the Carleman condition is the following one, which is a slight modification of a result presented by Billinglsley (1996, sec. 30, Theor. 30.1).

Theorem A.1: Let X be a r.v. with $\mu_r = E(X^r) < \infty$, $\forall r \in I\!N$. If the series

$$\sum_{r=1}^{\infty} \frac{\mu^r\, t^r}{r!} \tag{A.1}$$

has a positive radius of convergence, then the distribution of X is uniquely determined by its positive integer order moments. ■

We should note that, on one hand, the lack of compliance with the hypothesis of Theorem A.1 for a given r.v. does not imply that this r.v. is not uniquely determined by the set of its positive integer order moments, while, on the other hand, if the series in (A.1) has a positive radius of convergence, that is, if it is convergent for, say $t < \alpha\,(>0)$, then the m.g.f. of the r.v. X exists, with

$$M_X(t) \;=\; 1 + \sum_{r=1}^{\infty} \frac{\mu_r\, t^r}{r!}\,, \qquad (\text{for } t < \alpha)$$

so that through the unicity of the m.g.f. it is established that the r.v. X is the only one with positive integer moments ν_r.

The Carleman condition (Carleman, 1926) is usually referred as the result in the following Theorem.

Theorem A.2: Let $\mu_r = E(X^r) < \infty$. If

$$\sum_{r=1}^{\infty} \frac{1}{\mu_{2r}^{1/2r}} = +\infty$$

then the distribution of the r.v. X is uniquely determined by its positive integer order moments. ■

We should note that the result in Theorem A.1, although of the same type, is indeed stronger, since

$$\sum_{r=1}^{\infty} \frac{1}{\mu_{2r}^{1/2r}} = \infty \implies \sum_{r=1}^{\infty} \frac{\mu_{2r}\, t^r}{(2r)!} \ \text{has a positive radius of convergence,}$$

although both criteria place some restrictions on the rate of growth of the moments.

Another equivalent condition is the one presented in Theorem A.3.

Theorem A.3: Let $\mu_r^* = E\left(|X|^r\right)$. If

$$\limsup_{\tau\to\infty} \frac{\mu_\tau'^{1/r}}{} < \infty$$

then the distribution of X is determined by its positive integer order moments. ■

We should note that the condition in the hypothesis of Theorem A.3 implies indeed that the c.f. of the r.v. X has to be analytic (in any neighborhood of any $t \in I\!R$) (Feller, 1971, Chap. XV, sec. 4;

Shiryaev, 1996, p295), and as such that the distribution of the r.v. X is determined by its positive integer order moments.

A similar result to the one in Theorem A.3 is the following one.

Theorem A.4: Let $\mu_{2r} = E\left(X^{2r}\right)$. If

$$\limsup_{r\to\infty} \frac{\mu_{2r}^{1/2r}}{2r} < \infty$$

then the distribution of the r.v. X is determined by its positive integer order moments. ■

Wu (2002) presents a pair of simple and useful results, the first of them derived from Theorem A.1, by applying the D'Alembert criterium.

Theorem A.5: Let X be a r.v. with $E(X^r) < \infty$, $\forall r \in \mathbb{N}$. If

$$\lim_{r\to\infty} \frac{1}{r}\left|\frac{\mu_{r+1}}{\mu_r}\right| < \infty$$

then the distribution of the r.v. X is uniquely determined by its positive integer order moments. ■

And for discrete r.v.'s, derived from Theorem A.3 above we have the following result.

Theorem A.6: Let X be a discrete r.v. with support $\mathbb{N}$ and

$$p_k = P(X = k) \qquad \forall k \in \mathbb{N}$$

(and as such with $\sum_{k=1}^{\infty} p_k = 1$). If there is $\alpha \geq 1$ such that $p_k = O\left(e^{-k\alpha}\right)$, that is, such that $\forall k \in \mathbb{N}$, $p_k/e^{-k\alpha}$ remains bounded, then the distribution of the r.v. X is uniquely determined by the set of its positive integer order moments. ■

In a slightly different framework, Lin (1997) obtains four criteria which have the particularity of giving conditions both for a distribution to be determined and not determined by its integer order moments. The first two criteria are for r.v.'s with support $\mathbb{R}$, while the last two are for r.v.'s with support $\mathbb{R}^+$.

Theorem A.7: Let X be a r.v. with an absolutely continuous c.d.f. and p.d.f. $f(x) > 0$, $\forall x \in \mathbb{R}$. If $E(X^r) < \infty$, $\forall r \in \mathbb{N}$ and

$$\int_{-\infty}^{\infty} \frac{-\log f(x)}{1+x^2}\, dx < \infty\,, \tag{A.2}$$

then the distribution of the r.v. X is not uniquely determined by its positive integer order moments. ■

Theorem A.8: Let X be a r.v. with an absolutely continuous c.d.f. and p.d.f. $f(x) > 0$, $\forall x \in \mathbb{R}$, symmetrical about zero and differentiable in $\mathbb{R}$, such that $E(X^r) < \infty$, $\forall r \in \mathbb{N}$, with

$$f(x) \underset{x\to\infty}{\longrightarrow} 0\,, \qquad -x\,\frac{f'(x)}{f(x)} \underset{x\to\infty}{\longrightarrow} \infty$$

and

$$\int_{-\infty}^{\infty} \frac{-\log f(x)}{1+x^2}\, dx = \infty\,,$$

then the distribution of the r.v. X is uniquely determined by its positive integer order moments. ■

As pointed out by Lin (1997), a sufficient condition for (A.2) to hold is that the logarithmic mean function

$$g(t) = \frac{1}{2t}\int_{-t}^{t} |\log f(x)|\, dx \qquad (t \in \mathbb{R})$$

be bounded in $\mathbb{R}$ (see Paley and Wiener (1934, p128)).

Also Lin (1997) refers that while Theorem A.7 was already proved by Akhiezer (1965, p87), using a result from Krein (1945), he proves it using the Hardy space theory. Also as Lin (1997) shows in the proof of Theorem A.8, its hypothesis is a necessary condition for the Carleman condition to be satisfied.

Using the above two theorems it is possible to prove that while if X is a normally distributed r.v., its distribution is uniquely determined by its positive integer order moments, the distributions of X^{2n+1} ($n \in I\!\!N$) are not (Casella and Berger, 2002).

For r.v.'s in $I\!\!R^+$ we have the following two theorems.

Theorem A.9: Let X be a r.v. with an absolutely continuous c.d.f. and p.d.f. $f(x) > 0$, $\forall x \in I\!\!R^+$ and $f(x) = 0$, $\forall x \in I\!\!R_0^-$. If $E(X^r) < \infty$, $\forall r \in I\!\!N$ and

$$\int_0^\infty \frac{-\log f(x^2)}{1+x^2}\, dx \;<\; \infty\,,$$

then the distribution of the r.v. X is not uniquely determined by its positive integer order moments. ■

Theorem A.10: Let X be a r.v. with an absolutely continuous c.d.f. and p.d.f. $f(x) > 0$, $\forall x \in I\!\!R^+$ and $f(x) = 0$, $\forall x \in I\!\!R_0^-$, differentiable in $I\!\!R^+$, such that $E(X^r) < \infty$, $\forall r \in I\!\!N$, with

$$f(x) \underset{x\to\infty}{\longrightarrow} 0\,, \qquad -\,x\,\frac{f'(x)}{f(x)} \underset{x\to\infty}{\longrightarrow} \infty$$

and

$$\int_0^\infty \frac{-\log f(x^2)}{1+x^2}\, dx \;=\; \infty\,,$$

then the distribution of the r.v. X is uniquely determined by its positive integer order moments. ■

APPENDIX B

A brief note on some numerical issues

McCullagh (1994) claims that the m.g.f.^{s} of the r.v.'s X_n in (5) are numerically virtually undistinguishable (for $\alpha = 1/2$ and $n = 0$, $n = 1$) and Waller (1995), states that only the imaginary part of the c.f. helps in this numerical discrimination. However we should consider that although the numerical values of such functions may be quite close, they are analytically different for different values of n. Anyway, there is a clear scale problem when plotting the m.g.f.^{s} in (5). If instead we plot the difference between the values of the two m.g.f.^{s} and we take the plot far enough in terms of the absolute value of t it is no longer true that the two m.g.f.^{s} look the same. Actually, for example for $\alpha = 1/2$, $t = 9.9$ the two functions differ by something like $1.507163{\times}10^{12}$ while for $t = 12.7$ they differ by something like $1.343462{\times}10^{26}$. We should note that the above are also exactly the differences between the moments of order $h = 9.9$ and $h = 12.7$ for the r.v.'s X_0 and X_1 with p.d.f.^{s} in (1) for $\alpha = 1/2$ while if we take $\alpha = 1$ the corresponding differences will be roughly the double.

We should be aware that if we do not have enough precision in the computation process we may not be able to spot such differences since although being quite large in magnitude they are quite small when compared with the original values.

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